Unit 6: Quine-McCluskey

Quine-McCluskey: method which reduces a minterm expansion (standard sum of products) to obtain a minimum sum of products.

Main Steps

1. Eliminate as many literal as possible from each term by systematically applying the theorem XY +XY' = X. Resulting terms are prime implicants

2. Use a prime implicant chart to select a minimum set of prime implicants which, when ORed together, are equal to the function being simplified and which contain a minimum number of literals.

6.1 Determination of Prime Implicants

Note: The function must be in minterm format before proceeding

Express the minterms in binary notation and combine using XY +XY' = X.

Ex.

AB'CD'	+	AB'CD	=	AB'C
1010	+	1011	=	101-

To find all prime implicants, binary minterms are sorted into groups according to the number of 1's in each term.

Ex.

F (a, b, c, d) = Σ m (0, 1, 2, 5, 6, 7, 8, 9, 10, 14)

Grouping by the number of 1's

Group 0	0	0000
Group 1	1	0001
	2	0010
	8	1000
Group 2	5	0101
	6	0110
	9	1001
	10	1010
Group 3	7	0111
	14	1110

By XY +XY' = X, Two terms can be combined if they differ by exactly one variable. Each adjacent group can be compared but non-adjacent groups and the same groups can be ignored since they differ by more than one variable.

Note: Each time a term is combined with another the terms are checked off.

Group 0	0	0000
Group 1	1	0001
	2	0010
	8	1000
Group 2	5	0101
	6	0110
	9	1001
	10	1010
Group 3	7	0111
	14	1110

0,1	000-
0,2	00-0
0,8	-000
1,5	0-01
1,9	-001
2,6	0-10
2,10	-010
8,9	100-
8,10	10-0
5,7	01-1
6,7	011-
6,14	-110
10,14	1-10

Since the new groups also vary by only the number of 1's we can compare adjacent groups again. Compare terms that have dashes in corresponding places.

0,1,8,9	-00-
0,2,8,10	-0-0
0,8,1,9	-00-
0,8,2,10	-0-0
2,6,10,14	--10
2,10,6,14	--10

Comparing the above groups yields no further reduction.(the dashes don't line up in adjacent columns).

All unchecked terms are prime implicants; therefore the function can be wriiten as a standard sum of products.

F(a, b, c, d) =	a'c'd	+	a'bd	+	a'bc	+	b'c'	+	b'd'	+	cd'
	(1,5)		(5,7)		(6,7)		(0,1,8,9)		(0,2,8,10)		(2,6,10,14)

This represents the minimum number of literals but we need to use the consensus theorem to eliminate redundant terms.

$F(a, b, c, d) = a'bd + b'c' + cd'$

Implicant (Quine-McCluskey): Given a function F of n variable, a product term P is an implicacnt of F iff for every combination of values of the n variables for which P = 1, F is also equal to 1.

Prime Implicant (Quine-McCluskey): A prime implicant of a function F is a product term implicant which is no longer an implicant if any literal is deleted from it.

Note: A minimum sum of products expression contains only prime implicants but not necesssarily all the prime implicants.

6.2 The Prime Implicant Chart

To develop a prime implicants chart, minterms are listed across the top and prime implicants are on the left side. An X is placed at the intersection of a prime implicant and a minterm

If a minterm is covered by only 1 prime implicant then it is an essential prime implicant and must be included in the minimum sum of products. (Columns containing only one X)

Now cross out the rows of the essential prime implicants and each column that contains an X from the essential prime implicant. (Rows 1 and 3 columns 0, 1 ,2, 6, 8, 10)

Finally, choose a minimum set of prime implicant to cover the remaining minterms.

$$F = a'bd + b'c' + c'd'$$

Ex. Two or more minimum sum products

$$F(a, b, c) = \Sigma m(0, 1, 2, 5, 6, 7,)$$

Group 0	0	000
Group 1	1	001
	2	010
Group 2	5	101
	6	110
Group 3	7	111

0,1	00-
0,2	0-0
1,5	-01
2,6	-10
5,7	1-1
6,7	11-

Since there are no single X’s, must do trial and error approach.

For the 0 column, it is covered by (0,1) and (0,2)
Taking (0,1) - and crossing out the rows and columns

For the 2 column, it is covered by (0,2) and (2,6)
Taking (2,6) - and crossing out the rows and columns

Since the 5 and 7 column re all that remain
Taking (5,7) - completes the reduction

F =   a'b' + bc' + ac

To be sure this is the minimum solution we must look at the other options.

For the 0 column, it is covered by (0,1) and (0,2)
Taking (0,2) - and crossing out the rows and columns

For the 1 column, it is covered by (0,1) and (1,5)
Taking (1,5) - and crossing out the rows and columns

Since the 6 and 7 column re all that remain
Taking (6,7) - completes the reduction

F =   a'c' + b'c + ab

Since each solution has the same number of terms and literal both are minimum sum of products solutions.

Compared via a Karnaugh Map

BC \ A	0	1
00	1	
01	1	1
11		1
10	1	1

BC \ A	0	1
00	1	
01	1	1
11		1
10	1	1

6.3 Petrick's Method

A technique for determining all minimum sum of products solutions from a prime implicant chart.

For a prime implicant chart, label the rows P_1, P_2, etc. Where P_1 is true when the prime implicant in row P_1 is included in the solution.

For row 0, P_1 or P_2 are solutions $(P_1 + P_2)$

Describing all rows in terms of P

$$P = (P_1 + P_2)(P_1 + P_3)(P_2 + P_4)(P_3 + P_5)(P_4 + P_6)(P_5 + P_6) = 1$$

Simplifying with $(X + Y)(X + Z) = X + YZ$

$$P = (P_1 + P_2P_3)(P_4 + P_2P_6)(P_5 + P_3P_6) = 1$$

$P = (P_1P_4 + P_1P_2P_6 + P_2P_3P_4 + P_2P_3P_6)(P_5 + P_3P_6) = 1$

$P = P_1P_4P_5 + P_1P_2P_5P_6 + P_2P_3P_4P_5 + P_2P_3P_5P_6 + P_1P_3P_4P_6 + P_1P_2P_3P_6 + P_2P_3P_4P_6$

$+ P_2P_3P_6$

Eliminating redundant terms with X + XY = X
$(P_2P_3P_6)$

$P = P_1P_4P_5 + P_1P_2P_5P_6 + P_2P_3P_4P_5 + P_1P_3P_4P_6 + P_2P_3P_6$

Each product term is a viable solution but only two are the minimum solution $P_1P_4P_5$, and $P_2P_3P_6$.

Substituting back in for the values of P,

F = a'b' + bc' + ac   or   F = a'c' + b'c + ab

Summarizing Petrick's Method

1. Reduce the prime implicant chart by eliminating the essential prime implicant rows and corresponding columns

2. Label the rows of the reduced prime implicant chart P_1, P_2, etc.

3. Form a logic function P which is true when all columns are covered. P consists of product of sums terms, each having the form of $(P_{i0} + P_{i1} + \ldots)$ where P_{i0}, P_{i1} … represent the rows that cover column i.

4. Reduce P to a minimum sum of products by multiplying out and applying X + XY = X

5. Each term in the result represents a solution, that is, a set of rows which covers all the minterms in the table. To determine the minimum solutions find those terms which contain a minimum number of variables. Each of these terms represents a solution with a minimum number of prime implicants.

6. For each terms found in step 5, count the number of literals in each prime implicant. Choose the term or terms which correspond to the minimum number of literals and write out the corresponding sum of prime implicants.

6.4 Simplification of Incompletely Specified Functions

The don't cares are treated as minterms when finding the prime implicants in the Quine-McCluskey method.

However, when forming the prime implicant chart the don't cares are omitted.

Ex.

$$F(A, B, C, D) = \Sigma\, m(2, 3, 7, 9, 11, 13) + \Sigma\, d(1, 10, 15)$$

Note: The answer is completely specified although the original function was not and we also know the values for each don't care.

By inspection, d_{10} and d_{15} both appear in the final solution while d_1 does not.

For ABCD = 0001, F=0   For ABCD = 1011, F=1   For ABCD = 1111, F=0

6.5 Simplification Using Map-Entered Variables

Method to simplify functions using Karnaugh map techniques for more than four or five variables.

Ex. Using a 3 variable map for a 4 variable function

F(A,B,C,D) = A'B'C + A'BC + A'BC'D + ABCD + (AB'C)

First create the Map entering in D (since it occurs least) in the squares where it appears

BC \ A	0	1
00		
01	1	X
11	1	D
10	D	

Next simplify for D = 0 then for D =1

BC \ A	0	1
00		
01	1	X
11	1	
10		

F = A'C for D = 0

BC \ A	0	1
00		
01	X	X
11	X	1
10	1	

F = C + A'B for D = 1

$$F = A'C + D(C + A'B) = A'C + CD + A'BD$$

Note: The 1's in map D =1 are treated as don't cares since they were covered in D = 0

Compared to a 4 variable Karnaugh.

BC \ DA	00	01	11	10
00				
01	1	X	X	1
11	1		1	1
10				1

$$F = A'C + CD + A'BD$$

General Method:
If a variable P_i is placed in a square m_j of a map for function F, this means F = 1 when P_i = 1 and the variables are chosen so $m_j = 1$. For a map with P_1, P_2, …

$$F = MS_o + P_1MS_1 + P_2MS_2 \ldots$$

Where

MS_o is the minimum sum of products when $P_1 = P_2 = 0$

P_1MS_1 is the minimum sum of products when $P_1 = 1$, $P_2 = 0$ ($j \neq 1$) All 1's become all X's

P_2MS_2 is the minimum sum of products when $P_1 = 0$, $P_2 = 1$ ($j \neq 2$) All 1's become all X's

Ex. 4 variable map for a 6 variable function

$$G(A,B,C,D,E,F) = m_0 + m_2 + m_3 + Em_5 + Em_7 + Fm_9 + m_{11} + m_{15} + (\text{don't cares})$$

CD \ AB	00	01	11	10
00	1			
01	X	E	X	F
11	1	E	1	1
10	1			X

G

CD \ AB	00	01	11	10
00	1			
01	X		X	
11	1		1	1
10	1			X

E = F = 0

MS_o = A'B' + ACD

CD \ AB	00	01	11	10
00	X			
01	X	1	X	
11	X	1	X	X
10	X			X

E = 1, F = 0

P_1MS_1 = A'D

CD \ AB	00	01	11	10
00	X			
01	X		X	1
11	X		X	X
10	X			X

E = 0, F = 1

P_2MS_2 = AD

G = A'B' + ACD + EA'D + FAD

6.6 Conclusion

Four Methods for reduction to a minimum sum of products or minimum product of sums.

1. Algebraic – valuable when different forms of an expression are required

2. Karnaugh Map – easiest for 3 to 5 variables

3. Quine-McCluskey – Up to 15 variable but need a computer

4. Petrick’s Method – systematic solution from a prime implicant chart with many variables

Gating circuits

- Minimum sum of products yields a two level AND OR circuit
- Minimum product of sums yields a two level OR AND circuit

Looking Ahead:

Units 7 – 9 will discuss design techniques and multifunction circuit components