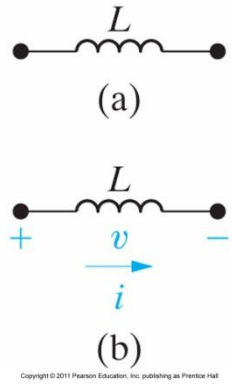
Chapter 6: Inductance, Capacitance, and Mutual Inductance

6.1 The Inductor

The figure shows the graphic symbol for an inductor, with an inductance of ***L*** in henrys (H).



The reference voltage and current for the inductor, following the passive sign convention.

$$v = L\frac{di}{dt}$$

Observations

1. If the current is constant the voltage across the inductor is zero. (short circuit)
2. Current cannot change instantaneously across the inductor

The current in an inductor

$$di = \frac{1}{L}vdt \xrightarrow{yields} i(t) = \frac{1}{L}\int_{t_0}^{t} vd\tau + i(t_0)$$

$$i(t) = \frac{1}{L}\int_{0}^{t} vd\tau + i(0)$$

Power and Energy

$$p = vi = L\frac{di}{dt}i = v\left[\frac{1}{L}\int_{0}^{t} vd\tau + i(t_0)\right]$$

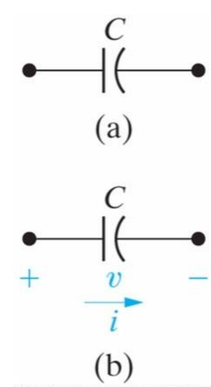
$$p = \frac{dw}{dt} = L\frac{di}{dt}i$$

$$w = \frac{1}{2}Li^2$$

Review Examples 6.1 – 6.3 and Assessment Problem 6.1

6.2 The Capacitor

The circuit symbol for a capacitor, with a Capacitance C in Farads (F)



Assigning reference voltage and current to the capacitor, following the passive sign convention.

$$i = C\frac{dv}{dt}$$

Observations

1. If the voltage across the terminals is constant the capacitor current is zero. (open circuit)
2. Voltage cannot change instantaneously across the terminals of the capacitors

The voltage across the capacitor

$$idt = Cdv \xrightarrow{yields} v(t) = \frac{1}{C}\int_{t_0}^{t} id\tau + v(t_0)$$

$$v(t) = \frac{1}{C}\int_{0}^{t} id\tau + v(0)$$

Power and Energy

$$p = vi = C\frac{dv}{dt}v = i\left[\frac{1}{C}\int_{0}^{t} id\tau + v(0)\right]$$

$$p = \frac{dw}{dt} = C\frac{dv}{dt}v$$

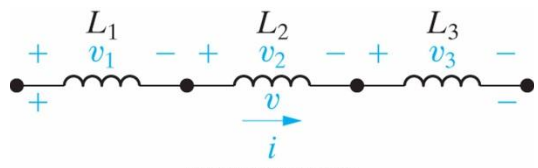
$$w = \frac{1}{2}Cv^2$$

Review Examples 6.4 & 6.5 and Assessment Problems 6.2 & 6.3

6.3 Series-Parallel Combinations of Inductances and Capacitances

Series inductors

$$v_1 = L_1\frac{di}{dt}, \qquad v_2 = L_2\frac{di}{dt}, \qquad v_3 = L_3\frac{di}{dt}$$



$$v = v_1 + v_2 + v_3 = (L_1 + L_2 + L_3)\frac{di}{dt}$$

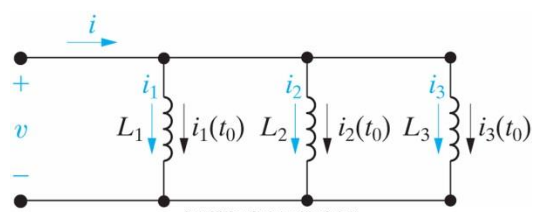
$$L_{eq} = L_1 + L_2 + L_3 + \cdots + L_n$$

Parallel inductors

$$i_1 = \frac{1}{L}\int_0^t vd\tau + i_1(0)$$

$$i_2 = \frac{1}{L}\int_0^t vd\tau + i_2(0)$$

$$i_3 = \frac{1}{L}\int_0^t vd\tau + i_3(0)$$



$$i = \left(\frac{1}{L_1} + \frac{1}{L_1} + \frac{1}{L_1}\right)\int_0^t vd\tau + i_1(0) + i_2(0) + i_3(0)$$

$$\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3} + \cdots + \frac{1}{L_n}$$

with initial current $i(t_0) = i_1(t_0) + i_2(t_0) + i_3(t_0) + \cdots + i_n(t_0)$

Series capacitors

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots + \frac{1}{C_n}$$

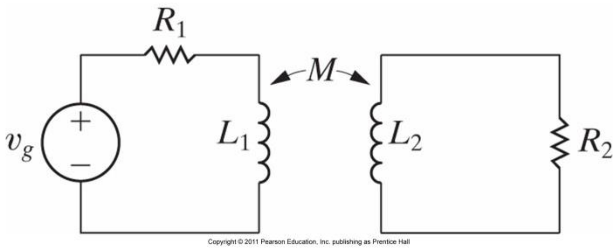
with initial voltage $v(t_0) = v_1(t_0) + v_2(t_0) + v_3(t_0) + \cdots + v_n(t_0)$

Parallel capacitors

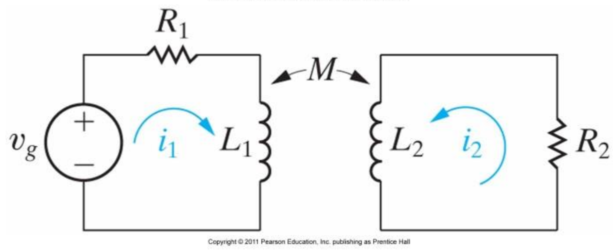
$$C_{eq} = C_1 + C_2 + C_3 + \cdots + C_n$$

Review Assessment Problems 6.4 & 6.5

6.4 **Mutual Inductance**



Two circuits linked by a magnetic field where the voltage induced in the second circuit can be related to the time varying current of the first.



Analyze these circuits using Mesh Analysis

Choose the reference current direction for i_1 & i_2 and sum the voltages in the closed paths

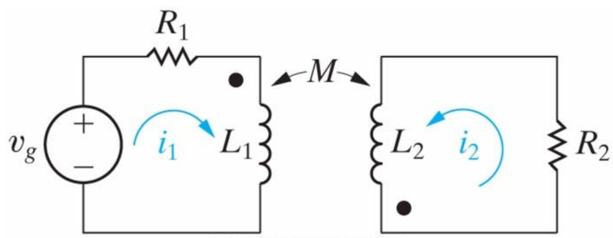
** Due to mutual inductance there are two voltages across each coil in the path **

1. *Self-induced voltage*: the product of the **self inductance** of the coil and the 1^{st} derivative of the current through **it**
2. *Mutually induced voltage*: the product of the **mutual inductance** of the coil**s** and the 1^{st} derivative of the **current through the other coil**

Polarity

Self induced voltage: a voltage drop in the direction of the current producing the voltage

Mutually induced voltage: is dependent on how the coils are wound



Dot convention: method to identify the polarities in a mutually induced voltage where a dot placed by one terminal of each winding

Rule: When the reference direction of the **current enters** the dotted terminal of a coil, the reference polarity of the voltage induced in the other coil is **positive** at its dotted terminal

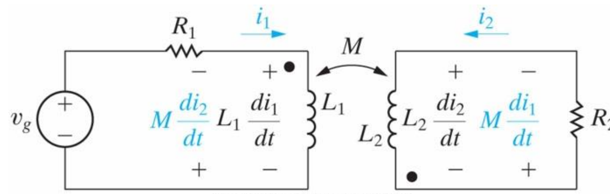
OR

When the reference direction of the **current leaves** the dotted terminal of a coil, the reference polarity of the voltage induced in the other coil is **negative** at its dotted terminal

Determining the mesh equations for each loop:

$$-v_g + i_1 R_1 + L_1 \frac{di_1}{dt} - M \frac{di_2}{dt} = 0$$

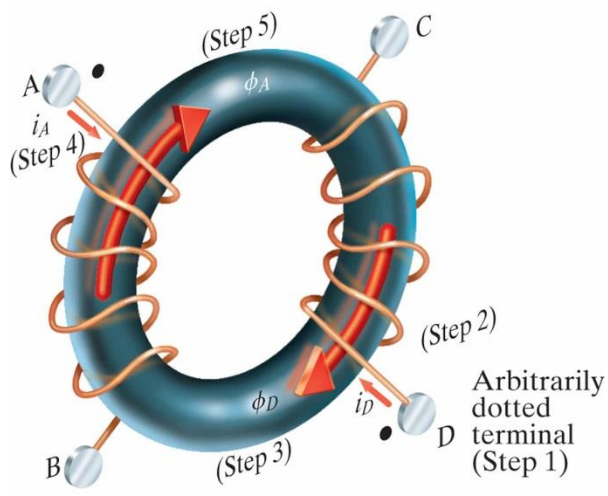
$$i_2 R_2 + L_2 \frac{di_2}{dt} - M \frac{di_1}{dt} = 0$$



Procedure for Determining Dot Markings

For known coil arrangements:

i. Arbitrarily mark one terminal of one coil (D)
ii. Assign a current into that terminal (i_D)
iii. Use the right-hand rule to determine the direction of the magnetic flux inside the coils
iv. Arbitrarily pick one terminal (A) of the other coil and assign a current into it (i_A)
v. Use the right-hand rule to determine the direction of the magnetic flux from this current inside the coils
vi. Compare the two fluxes directions: If the direction is the same place a dot at the referenced terminal (A) if they are opposite select the opposite terminal (B).



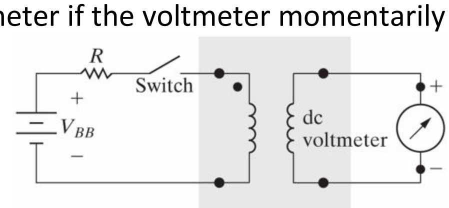
Experimentally (unknown):

Connect a DC voltage, resistor and switch to one coil pair and a DC voltmeter to the other.

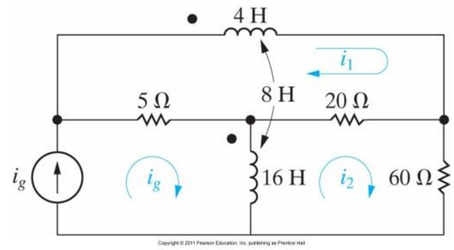
- The first polarity mark goes at the terminal connected to the resistor and switch.

When the switch is closed…

- The second polarity mark goes on the terminal connected to the **positive terminal** of the voltmeter if the voltmeter momentarily deflects **upscale**
- The second polarity mark goes on the terminal connected to the **negative terminal** of the voltmeter if the voltmeter momentarily deflects **downscale**



Example 6.6 - *page 192*
Mesh equations



i. $4\frac{di_1}{dt} + 8\frac{d}{dt}(i_g - i_2) + 20(i_1 - i_2) + 5(i_1 - i_g) = 0$

ii. $20(i_2 - i_1) + 60i_2 + 16\frac{d}{dt}(i_2 - i_g) - 8\frac{di_1}{dt} = 0$

Checking validity:

Given no stored energy at $t = 0$ and $i_g = 16 - 16e^{-5t}$

$$i_1 = 4 + 64e^{-5t} - 68e^{-4t}A \quad \text{and} \quad i_2 = 1 - 52e^{-5t} + 51e^{-4t}A$$

Initial

$$i_1(0) = 4 + 64 - 68 = 0 \quad \text{and} \quad i_2(0) = 1 - 52 + 51 = 0$$

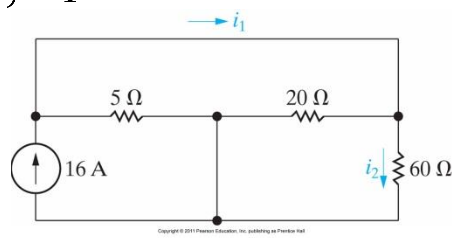
Final

$$i_1(\infty) = 4 \quad \text{and} \quad i_2(\infty) = 1$$

Comparing to the final circuit

$$20(i_1 - i_2) + 5(i_1 - i_g) = 0$$

$$20(i_2 - i_1) + 60i_2 = 0$$



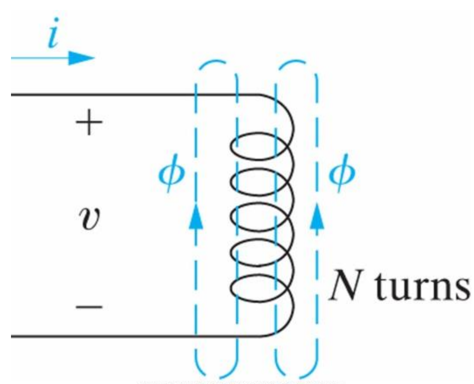
Review Assessment Problem 6.6

6.5 A Closer Look at Mutual Inductance

Self Inductance:

Faraday's Law: $\boldsymbol{v} = \frac{\boldsymbol{d\lambda}}{\boldsymbol{dt}}$

where λ is the flux linkage in weber-turns



The flux can be written:

$$\boldsymbol{\lambda = N\phi}$$

$\boldsymbol{\phi}$ represents the magnetic field in webers; its direction in the coil is determined by the right-hand rule.

$$\boldsymbol{\phi = \wp N i}$$

where $\wp$ is the permeance of the space occupied by the flux; it describes the magnetic properties of the space.

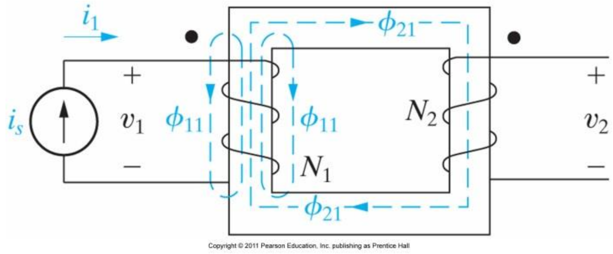
Assuming the space occupied by the flux is nonmagnetic yields the following through substitution:

$$\boldsymbol{v = \frac{d\lambda}{dt} = \frac{d(N\phi)}{dt} = N\frac{d\phi}{dt} = N\frac{d(\wp N i)}{dt} = N^2\wp\frac{di}{dt} = L\frac{di}{dt}}$$

Therefore self inductance is proportional to the square of the number of turns.

Mutual Inductance:

Analyzing the following circuit containing two magnetically coupled coils with a current source on coil 1 and coil 2 open:



The total flux linking in coil 1

$$\boldsymbol{\phi_1 = \phi_{11} + \phi_{21}}$$

Where

$$\boldsymbol{\phi_1} = \wp_1 N_1 i_1; \qquad \boldsymbol{\phi_{11}} = \wp_{11} N_1 i_1; \qquad \boldsymbol{\phi_{21} = \wp_{21} N_1 i_1;}$$

And

$$\wp_1 = \wp_{11} + \wp_{21}$$

Again using Faraday's Law:

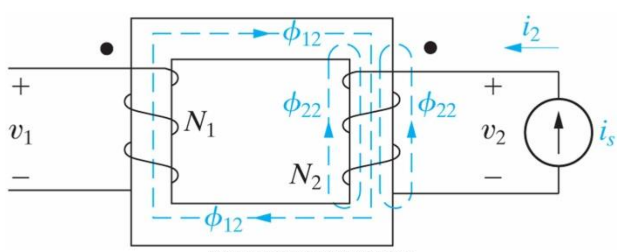
$$v_1 = \frac{d\lambda_1}{dt} = \frac{d(N_1\phi_1)}{dt} = N_1 \frac{d(\phi_{11} + \phi_{21})}{dt} = N_1^2(\wp_{11} + \wp_{21})\frac{di_1}{dt} = N_1^2 \wp_1 \frac{di_1}{dt} = L_1 \frac{di_1}{dt}$$

$$v_2 = \frac{d\lambda_2}{dt} = \frac{d(N_2\phi_{21})}{dt} = N_2 \frac{d(\wp_{21} N_1 i_1)}{dt} = N_2 N_1 \wp_{21} \frac{di_1}{dt}$$

Where the coefficients of the derivative of v_1 are the self-inductance and v_2 are the mutual inductance.

Thus $\quad M_{21} = N_2 N_1 \wp_{21} \quad$ and $\quad v_2 = M_{21}\frac{di_1}{dt}$

Now, analyzing the circuit containing two magnetically coupled coils with a current source on coil 2 and coil 1 open:



The total flux linking in coil 2

$$\boldsymbol{\phi_2 = \phi_{22} + \phi_{12}}$$

Where

$$\boldsymbol{\phi_2} = \wp_2 N_2 i_2; \qquad \boldsymbol{\phi_{22}} = \wp_{22} N_2 i_2; \qquad \boldsymbol{\phi_{12} = \wp_{12} N_2 i_2;}$$

Finding the voltages:

$$v_2 = \frac{d\lambda_2}{dt} = N_2^2 \wp_2 \frac{di_2}{dt} = L_2 \frac{di_2}{dt}$$

$$v_1 = \frac{d\lambda_1}{dt} = \frac{d(N_1\phi_{12})}{dt} = N_2 N_1 \wp_{12} \frac{di_2}{dt}$$

Thus

$$\boldsymbol{M_{12} = N_2 N_1 \wp_{12}}$$

For nonmagnetic material the permeances are equal and hence; $\boldsymbol{M_{12} = M_{21} = M}$ therefore for linear circuits with only **2** magnetically coupled coils subscripts are not required for the mutual inductance.

Mutual Inductance in Terms of Self Inductance

From the derived voltage equation we can write:

$\boldsymbol{L_1 = N_1^2 \wp_1}$ and $\boldsymbol{L_2 = N_2^2 \wp_2;}$

combining

$$\boldsymbol{L_1 L_2 = N_1^2 \wp_1 N_2^2 \wp_2 = N_1^2 N_2^2 (\wp_{11} + \wp_{21})(\wp_{22} + \wp_{12})}$$

In linear circuits: $\boldsymbol{\wp_{12} = \wp_{21}}$ and can be factored out of each bracket

$$\boldsymbol{L_1 L_2 = N_1^2 N_2^2 \wp_{12}^2 \left(1 + \frac{\wp_{11}}{\wp_{12}}\right)\left(1 + \frac{\wp_{22}}{\wp_{12}}\right)}$$

For $\boldsymbol{M_{12} = N_2 N_1 \wp_{12} = M}$ and replacing the permeance terms with a constant:

$$\frac{1}{k^2} = \boldsymbol{\left(1 + \frac{\wp_{11}}{\wp_{12}}\right)\left(1 + \frac{\wp_{22}}{\wp_{12}}\right)}$$

Yields $\boldsymbol{L_1 L_2 = \frac{1}{k^2} M^2}$ rewriting gives $\boldsymbol{M^2 = k^2 L_1 L_2}$

Or

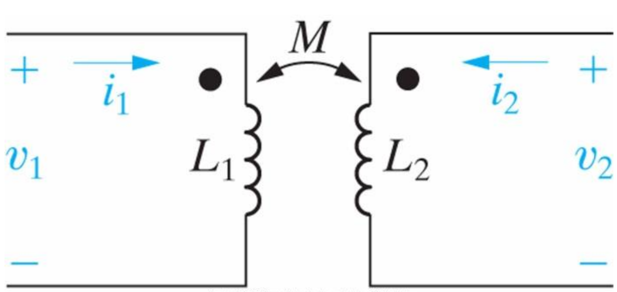
$\boldsymbol{M = k\sqrt{L_1 L_2}}$ Where **k** is called the **coefficient of coupling** and 0 ≤ k ≥ 1

When k = 0; there is no shared flux $\boldsymbol{\wp_{12} = 0}$ and Thus M = 0

When k =1; all the flux links the coils $\boldsymbol{\wp_{11} = \wp_{22} = 0}$ which is an ideal state

Energy Calculations

Deriving the total energy stored in a magnetic fields of a pair of linearly coupled coils



For $i_1 = i_2 = 0$ the power is zero

For i_1 increasing to I_1 ; $i_2 = 0$

$$p = v_1 i_1 = i_1 L_1 \frac{di_1}{dt} = \frac{dw}{dt}$$

$$dw = i_1 L_1 \, di_1$$

Integrating both sides

$$W = \frac{L_1 I_1^2}{2}$$

For $i_1 = I_1;\ i_2$increasing to I_2

The voltage induce in coil 1 from 2 is $M_{12}\frac{di_2}{dt}$ and the voltage induce in coil 2 from 1 is 0

$$p = I_2 M_{12}\frac{di_2}{dt} + v_2 i_2$$

This is similar to the above power equation with the addition of the mutual component

$$W = \frac{L_2 I_2^2}{2} + I_1 I_2 M_{12}$$

For $i_1 = I_1\ and\ i_2 = I_2$

Total Energy stored is

$$W = \frac{L_1 I_1^2}{2} + \frac{L_2 I_2^2}{2} + I_1 I_2 M_{12}$$

Starting with $i_2$increasing to I_2 first and going through the steps yields:

$$W = \frac{L_1 I_1^2}{2} + \frac{L_2 I_2^2}{2} + I_1 I_2 M_{21}$$

Since $M_{12} = M_{21}$ in a linear coupling and applying the solution to the instantaneous currents results in

General form

$$\boldsymbol{w(t) = \frac{L_1 i_1^2}{2} + \frac{L_2 i_2^2}{2} \pm i_1 i_2 M}$$

Note: The sign for the energy from the mutual inductance is positive if **both** currents enter (or leave) the dotted terminals. The sign is negative when one current leaves the terminal and the other enters the terminal.