Chapter 13: The Laplace Transform in Circuit Analysis

13.1 Circuit Elements in the *s*-Domain

Creating an s-domain equivalent circuit requires developing the time domain circuit and transforming it to the s-domain

Resistors:

Time-domain

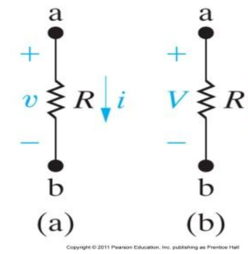
$$v = iR$$

s-domain

$$V = IR$$

Where

$$V = \mathcal{L}\{v\} \ and \ I = \mathcal{L}\{i\}$$



Inductors: (initial current I_0)

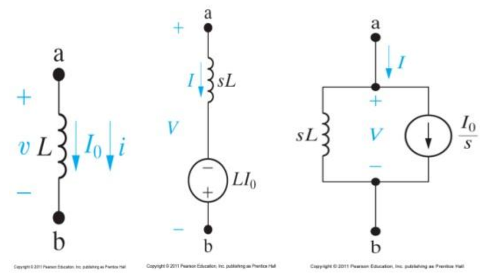
Time-domain:

$$v = L\frac{di}{dt}$$

s-domain:

Configuration #1: an impedance sL in series with an independent voltage source LI_0

$$V = L[sI - i(0^-)] = sLI - LI_0$$



Configuration #2: an impedance sL in parallel with an independent current source I_0/s

$$I = \frac{V + LI_0}{sL} = \frac{V}{sL} + \frac{I_0}{s}$$

If the initial current is zero the s-domain circuit for both representations simplifies to just the impedance sL.

Capacitors:

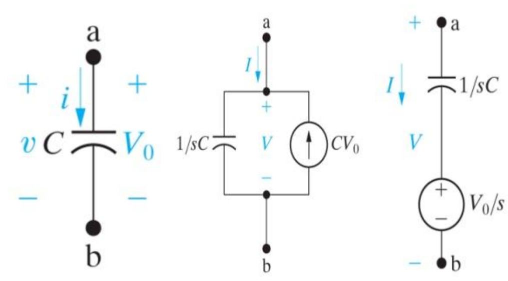
Time-domain

$$i = C\frac{dv}{dt}$$

s-domain

Configuration #1: an admittance sC in parallel with an independent current source CV_0

$$I = C[sV - v(0^-)] = sCV - LV_0$$



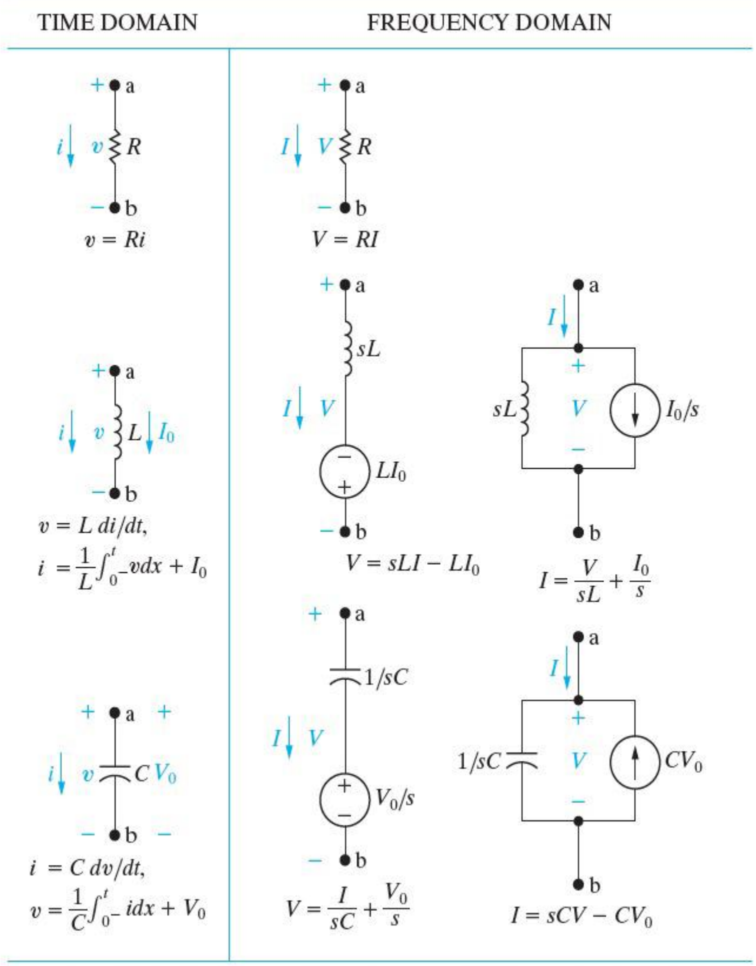
Configuration #2: an admittance sC in series with an independent voltage source V_0/s

$$V = \frac{I}{sC} + \frac{V_0}{s}$$

If the initial voltage is zero the s-domain circuit for both representations simplifies to just the admittance sC.

Note: An important first step in problem-solving will be to choose the correct s-domain series or parallel equivalent circuits to model your circuit.

TABLE 13.1 Summary of the *s*-Domain Equivalent Circuits



13.2 Circuit Analysis in the *s*-Domain

Before performing circuit analysis on s-domain circuits, it is necessary to understand the basic concepts.

If there is no energy stored in an inductor or capacitor then for all elements

$$V = ZI$$

With impedances;

Resistor R Inductor sL *Capacitor* $1/_{sC}$

Admittances;

Resistor $1/_{R}$ Inductor $1/_{sL}$ *Capacitor* sC

The following rules and techniques apply to the s-domain

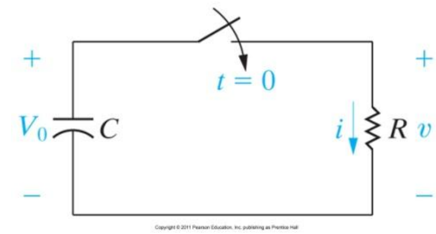
- Series and parallel impedances
- $\Delta - Y$conversions
- Kirchhoff's Laws
- Node and Mesh analysis
- Thevenin-Norton equivalents

13.3 Applications

Since the equations in the s-domain rely on algebraic manipulation rather than differential equations as in the time domain it should prove easier to work in the s-domain.

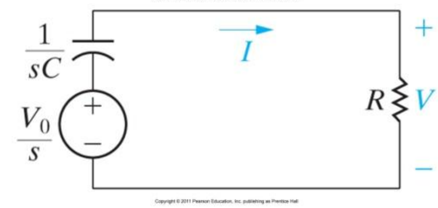
The Natural Response of an RC Circuit

Assuming an initial charge of V_0 on the capacitor:



Solving for i:

First we need to determine the s-domain circuit that best fits our need: (series equivalent for current)



Summing the voltages around the mesh:

$$\frac{V_0}{s} = \frac{1}{sC} I + RI$$

$$I = \frac{CV_0}{RCs + 1} = \frac{\frac{V_0}{R}}{s + {}^{1}/_{RC}}$$

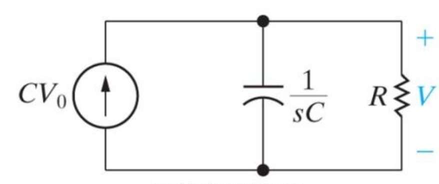
Taking the inverse transform:

$$\mathcal{L}^{-1}\{I\} = i = \frac{V_0}{R} e^{-t/_{RC}} u(t)$$

To solve for *v*:

$$v = Ri \;\therefore v = V_0 e^{-t/_{RC}} u(t)$$

Repeating the problem by solving for v:



The s-domain circuit that best fits our need is a parallel equivalent for voltage.

Nodal analysis:

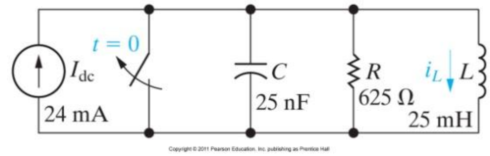
$$\frac{V}{R} + sCV = CV_0 \quad \rightarrow \quad V = \frac{V_0}{s + {}^{1}/_{RC}}$$

$$\mathcal{L}^{-1}\{V\} = v = V_0 e^{-t/_{RC}} u(t) = V_0 e^{-t/_{\tau}} u(t)$$

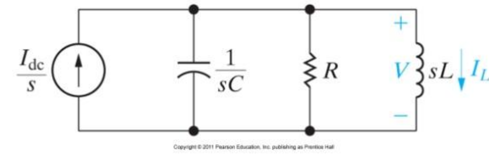
Again the voltage determined was the same but different equivalent circuits were used depending on the desired response to be determined.

The Step Response of a Parallel Circuit

For the parallel RLC circuit shown find I_L:



Create an equivalent s-domain circuit…



Note: the source can be modeled in the s-domain since it will appear as a step the moment the switch is closed

$$\mathcal{L}\{I_{dc}u(t)\} = I_{dc}/s$$

The current can be determined once the voltage is known

$$sCV + \frac{V}{R} + \frac{V}{sL} = \frac{I_{dc}}{s}$$

Solving for V

$$V = \frac{\frac{I_{dc}}{C}}{s^2 + \frac{1}{RC}s + \frac{1}{LC}}$$

Solving for I_L

$$I_L = \frac{V}{sL} = \frac{\frac{I_{dc}}{LC}}{s\left(s^2 + \frac{1}{RC}s + \frac{1}{LC}\right)}$$

Substituting in values

$$I_L = \frac{384 \times 10^5}{s(s^2 + 64000s + 16 \times 10^8)} = \frac{384 \times 10^5}{s(s + 32000 - j24000)(s + 32000 + j24000)}$$

Checking the final value theorem

$$\lim_{s\to 0} sI_L = \frac{384 \times 10^5}{(16 \times 10^8)} = 24mA$$

Partial Fractions

$$I_L = \frac{K_1}{s} + \frac{K_2}{(s + 32000 - j24000)} + \frac{K_2^*}{(s + 32000 + j24000)}$$

Solving for the coefficients

$$K_1 = \frac{384 \times 10^5}{16 \times 10^8} = 24 \times 10^{-3}$$

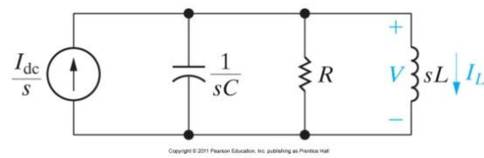
$$K_2 = \frac{384 \times 10^5}{(-32000 + j24000)j48000} = 20 \times 10^{-3}\angle 126.87°$$

Taking the inverse transform

$$i_L = [24 + 40e^{-32000t}\cos(24000t + 126.87°)]u(t)$$

We could then check the initial and final value theorem to confirm that the i_L solution satisfied the given initial conditions and final behavior.

The Transient Response of a Parallel RLC Circuit



Replacing the DC current source in the previous problem with a sinusoidal source

$$i_g = I_m \cos \omega t.$$

Where $I_m = 24mA \quad and \quad \omega = 40000 \ rad/s$

$$\mathcal{L}\{i_g u(t)\} = I_g = \frac{sI_m}{s^2 + \omega^2}$$

Finding the new voltage expression

$$V = \frac{\left(\frac{I_m}{C}\right)^2 s^2}{(s^2 + \omega^2)\left(s^2 + \frac{1}{RC}s + \frac{1}{LC}\right)}$$

Solving for I_L

$$I_L = \frac{V}{sL} = \frac{\left(\frac{I_m}{LC}\right)^2 s}{(s^2 + \omega^2)\left(s^2 + \frac{1}{RC}s + \frac{1}{LC}\right)}$$

Substituting in values

$$I_L = \frac{384 \times 10^5 s}{(s^2 + 16 \times 10^8)(s^2 + 64000s + 16 \times 10^8)}$$

$$I_L = \frac{384 \times 10^5 s}{(s - j40000)(s + j40000)(s + 32000 - j24000)(s + 32000 + j24000)}$$

Partial Fractions

$$I_L = \frac{K_1}{(s - j40000)} + \frac{K_1^*}{(s + j40000)} + \frac{K_2}{(s + 32000 - j24000)} + \frac{K_2^*}{(s + 32000 + j24000)}$$

Solving for the coefficients

$$K_1 = \frac{384 \times 10^5 (j40000)}{j80000(32000 + j16000)(32000 + j64000)} = 7.5 \times 10^{-3} \angle -90°$$

$$K_2 = \frac{384 \times 10^5 (-32000 + j24000)}{(-32000 - j16000)(-32000 + j64000)j48000} = 12.5 \times 10^{-3} \angle 90°$$

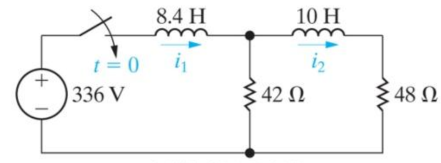
Taking the inverse transform

$$i_L = [15 \cos(40000t - 90°) + 25e^{-32000t} \cos(24000t + 90°)]u(t)$$
$$= [15 \sin 40000t - 25e^{-32000t} \sin 24000t]u(t)$$

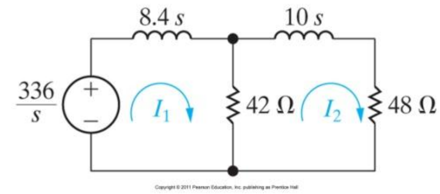
Checking the initial and final value will confirm if the solution satisfies the behavior

The Step Response of a Multiple Mesh Circuit

Previously we avoided circuits with multiple mesh currents or node voltage due to the need to solve simultaneous differential equations.



Since Laplace allows for algebraic manipulation we can solve a circuit like the one to the right.



First find the s-domain equivalent circuit… then write the necessary mesh or node equations.

$$\frac{336}{s} = (42 + 8.4s)I_1 - 42I_2$$

$$0 = -42I_1 + (90 + 10s)I_2$$

Using Cramer's rule to solve

$$\Delta = \begin{vmatrix} 42 + 8.4s & -42 \\ -42 & 90 + 10s \end{vmatrix} = 84(s^2 + 14s + 24) = 84(s + 2)(s + 12)$$

$$N_1 = \begin{vmatrix} 336/s & -42 \\ 0 & 90 + 10s \end{vmatrix} = \frac{3360(s + 9)}{s}$$

$$N_2 = \begin{vmatrix} 42 + 8.4s & 336/s \\ -42 & 0 \end{vmatrix} = \frac{14112}{s}$$

$$I_1 = \frac{N_1}{\Delta} = \frac{40(s + 9)}{s(s + 2)(s + 12)}$$

$$I_2 = \frac{N_2}{\Delta} = \frac{168}{s(s + 2)(s + 12)}$$

Expanding into partial fractions

$$I_1 = \frac{15}{s} - \frac{14}{s + 2} - \frac{1}{s + 12}$$

$$I_2 = \frac{7}{s} - \frac{8.4}{s + 2} + \frac{1.4}{s + 12}$$

Taking the inverse transform

$$i_1 = (15 - 14e^{-2t} - e^{-12t})u(t)$$

$$i_2 = (7 - 8.4e^{-2t} + 1.4e^{-12t})u(t)$$

Again checking for validity, since there is no stored energy at $t = 0^-$ both currents should be zero. (which is the case)

Evaluating at ∞,

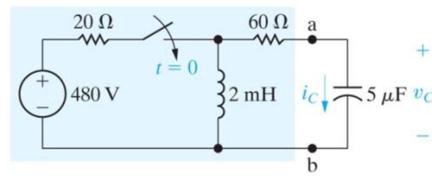
$$i_1 = 15A \quad and \quad i_2 = 7A$$

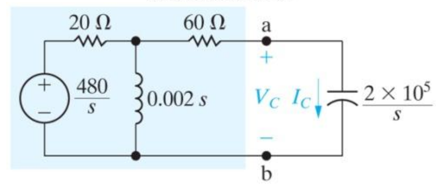
The voltage drop across the 42Ω resistor:

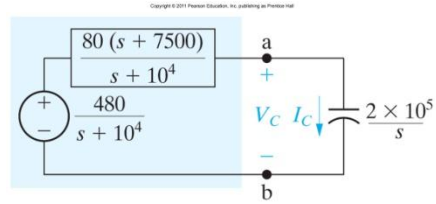
$$v = 42(i_1 - i_2) = 42(8 - 5.6e^{-2t} + 1.8e^{-12t})u(t)$$

The Use of Thevenin's Equivalent

To find i_C in the following circuit, first convert to the equivalent s-domain circuit.







The Thevenin voltage is the open circuit voltage across terminals a and b.(Open circuit conditions means no voltage across the 60 ohm resistor)

$$V_{th} = \frac{(480/s)0.002s}{20 + 0.002s} = \frac{480}{s + 10000}$$

The Thevenin impedance is the equivalent impedance seen at the terminals with the source shorted.

$$Z_{th} = 60 + \frac{(20)0.002s}{20 + 0.002s} = \frac{80(s + 7500)}{s + 10000}$$

Now a Thevenin equivalent circuit can be created and i_C can be determined

$$I_C = \frac{V_{th}}{Z_{th} + Z_C} = \frac{480}{s + 10000} * \frac{1}{\left[\frac{80(s + 7500)}{(s + 10000)}\right] + \left[\frac{2 \times 10^5}{s}\right]}$$

$$= \frac{6s}{s^2 + 10000s + 25 \times 10^6} = \frac{6s}{(s + 5000)^2}$$

Partial fraction

$$I_C = \frac{-30000}{(s + 5000)^2} + \frac{6}{s + 5000}$$

Taking the inverse transform

$$i_C = (-30000te^{-5000t} + 6e^{-5000t})u(t)$$

Again checking for validity is necessary.

$$i_C(0) = 6A$$

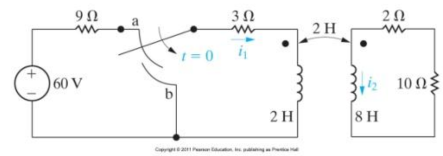
If the voltage v_C were desired we could integrate the current times the capacitance or perform the s-domain equivalent and then transform to the time domain

$$V_C = \frac{1}{sC} I_C = \frac{12 \times 10^5}{(s + 5000)^2}$$

$$v_C = 12 \times 10^5 te^{-5000t}u(t)$$

A Circuit with Mutual Inductance

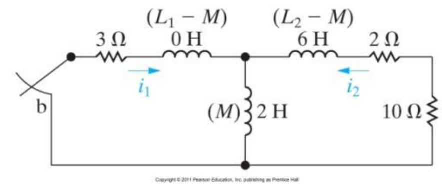
When analyzing a circuit with mutual inductance it is necessary to first transform into the T-equivalent circuit.



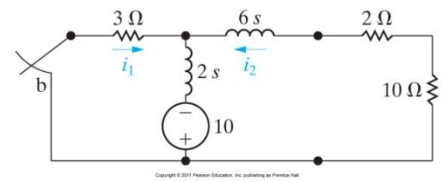
The left branch of the T is $L_1 - M$

The right branch $L_2 - M$

The base is just M.



Once the T-equivalent circuit is complete it circuit can be transformed to the s-domain.



Note: $i_1(0^-) = \frac{60}{12} = 5A$ and $i_2(0^-) = 0$

Solving for the two currents

$$(3 + 2s)I_1 + 2sI_2 = 10$$

$$2sI_1 + (12 + 8s)I_2 += 10$$

Solving for I_2

$$I_2 = \frac{2.5}{(s+1)(s+3)} = \frac{1.25}{s+1} - \frac{1.25}{s+3}$$

Therfore

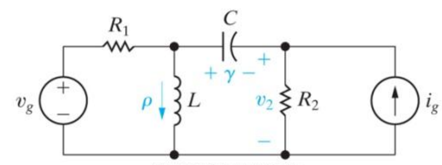
$$i_2 = (1.25e^{-t} - 1.25e^{-3t})u(t)$$

Checking for validity at and $i_2(0^-)$ shows it is zero as predicted

The Use of Superposition

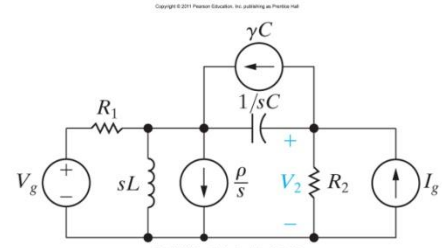
This allows a response to be divided into components that are identified with a particular source and initial conditions.

When the switches are closed on the following circuit assume the initial current in the inductor is ρ and voltage in the capacitor is γ



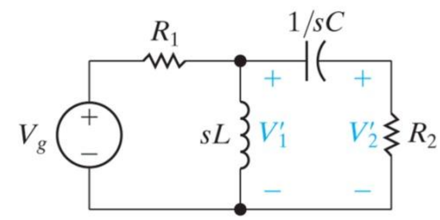
If the desired response is v_2

Find the equivalent s-domain circuit using the parallel equivalents for the capacitor and inductor since the desired response is a voltage.



Now solve by calculating the component of v_2 due to each source and then sum them together.

Solving for V_g alone requires opening the other current sources and analyzing the remaining circuit. (Note: the desired voltages are shown with a prime to indicate they are due to V_g.)



Solving for the two equations

$$\left(\frac{1}{R_1}+\frac{1}{sL}+sC\right)V_1'-sCV_2'=\frac{V_g}{R_1}$$

$$-sCV_1'+\left(\frac{1}{R_2}+sC\right)V_2'=0$$

To facilitate the remaining circuits use:

$$Y_{11}=\frac{1}{R_1}+\frac{1}{sL}+sC$$

$$Y_{12}=-sC$$

$$Y_{22}=\frac{1}{R_2}+sC$$

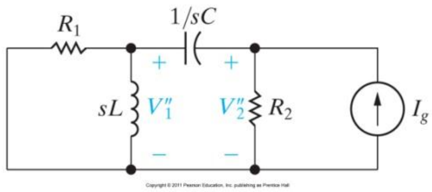
Rewriting the original equations

$$Y_{11}V_1'+Y_{12}V_2'=\frac{V_g}{R_1}\quad and \quad Y_{12}V_1'+Y_{22}V_2'=0$$

Solving for V_2'

$$V_2'=\frac{-\frac{Y_{12}}{R_1}}{Y_{11}Y_{22}-Y_{12}^2}V_g$$

Now the analysis must be performed for I_g alone; create a circuit with the current sources open and voltages shorted. (use double primes on the voltage to indicate it is due to I_g)



Solving for the equations

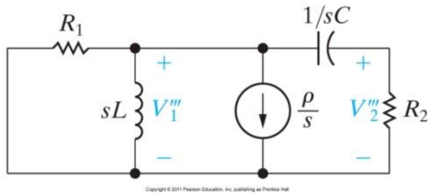
$$Y_{11}V_1''+Y_{12}V_2''=0$$

$$Y_{12}V_1''+Y_{22}V_2''=I_g$$

Solving for V_2''

$$V_2''=\frac{Y_{11}}{Y_{11}Y_{22}-Y_{12}^2}I_g$$

Now solving for V_2 due to the initial energy in the inductor. (use triple primes on the voltages)



Solving for the equations

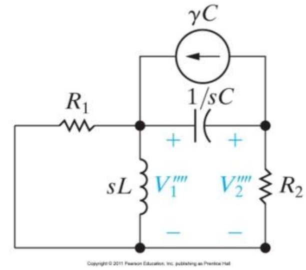
$$Y_{11}V_1'''+Y_{12}V_2'''=-\frac{\rho}{s}$$

$$Y_{12}V_1'''+Y_{22}V_2'''=0$$

Solving for V_2'''

$$V_2''' = \frac{\frac{Y_{12}}{s}}{Y_{11}Y_{22} - Y_{12}^2}\rho$$

Finally find the final component of V_2 due to the initial charge on the capacitor. (use four primes to indicate these voltages)



Solving for the equations

$$Y_{11}V_1'''' + Y_{12}V_2'''' = \gamma C$$

$$Y_{12}V_1'''' + Y_{22}V_2'''' = -\gamma C$$

Solving for V_2''''

$$V_2'''' = \frac{-(Y_{11} + Y_{12})C}{Y_{11}Y_{22} - Y_{12}^2}\gamma$$

Solving for V_2

$$V_2 = V_2' + V_2'' + V_2''' + V_2''''$$

$$V_2 = \frac{-\frac{Y_{12}}{R_1}}{Y_{11}Y_{22} - Y_{12}^2}V_g + \frac{Y_{11}}{Y_{11}Y_{22} - Y_{12}^2}I_g + \frac{\frac{Y_{12}}{s}}{Y_{11}Y_{22} - Y_{12}^2}\rho + \frac{-C(Y_{11} + Y_{12})}{Y_{11}Y_{22} - Y_{12}^2}\gamma$$

We could also solve for V_2 without superposition by just writing the node equations

$$Y_{11}V_1 + Y_{12}V_2 = \frac{V_g}{R_1} + \gamma C - \frac{\rho}{s}$$

$$Y_{12}V_1 + Y_{22}V_2 = I_g - \gamma C$$

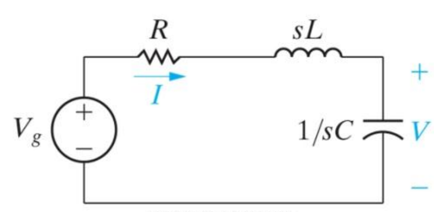
13.4 The Transfer Function

Transfer Function: the s-domain ratio of the Laplace transform of the output (response) to the Laplace transform of the input (source)

$$H(s) = \frac{\mathcal{L}\{output\}}{\mathcal{L}\{input\}} = \frac{Y(s)}{X(s)}$$

Example. Finding the transfer function of an RLC circuit

If the current is the desired output:



$$H(s) = \frac{I}{V_g} = \frac{1}{R + sL + {}^1\!/_{sC}} = \frac{sC}{s^2LC + RCs + 1}$$

If the voltage is the desired output:

$$H(s) = \frac{V}{V_g} = \frac{{}^{1}/_{sC}}{R + sL + {}^{1}/_{sC}} = \frac{1}{s^2LC + RCs + 1}$$

Note: Since a circuit may have multiple sources and the response of interest may vary a single circuit can generate multiple transfer functions.

(Review Example 13.1)

The Location of Poles and Zeros of H(s)

- H(s) is always a rational function of s.
- Complex poles and zeros appear in conjugated pairs
- The poles of H(s) must lie in the left-half of the s-plane
- The zeros of H(s) can lie in either half of the s-plane

13.5 The Transfer Function in Partial Fraction Expansions

$$Y(s) = H(s)X(s)$$

From the sum of partial fractions;

- The terms generated from the poles of H(s) describe the transient component of the response.
- The terms generated from the poles of X(s) describe the steady-state component of the response. (response after transients have become negligible)

(Review Example 13.2)

Observations of the Use of H(s) in Circuit Analysis

If the time it takes to reach the maximum value of the circuit is long compared to its time constants, the solution assuming an unbounded ramp is valid for a finite time

Effects of delays on the response

$$\mathcal{L}\{x(t-a)u(t-a)\} = e^{-as}X(s)$$

Then

$$Y(s) = H(s)X(s)e^{-as}$$

If $y(t) = \mathcal{L}^{-1}\{Y(s)\}$

$$y(t-a)u(t-a) = \mathcal{L}^{-1}\{H(s)X(s)e^{-as})\}$$

Thus delaying the input by *a* will delay the response by *a*. A circuit with this relationship is said to be **time invariant.**

If a unit impulse drives the circuit, the response of the circuit equals the inverse transform of the transfer function.

If $y(t) = \delta(t)$ then $X(s) = 1$

$$Y(s) = H(s)$$

Therefore

$$y(t) = h(t)$$

13.6 The Transfer Function and the Convolution Integral

13.7 The Transfer Function and the Steady-State Sinusoidal Response

13.8 The Impulse Function in Circuit Analysis