Chapter 16: Fourier Series

16.1 Fourier Series Analysis: An Overview

A periodic function can be represented by an infinite sum of sine and cosine functions that are harmonically related:

$$f(t) = a_v + \sum_{n=1}^{\infty} a_n \cos n\omega_o t + b_n \sin n\omega_o t$$

<u>Fourier Coefficients:</u> $a_v;\ a_n;\ b_n$ are calculated from $f(t)$

<u>Fundamental Frequency:</u> $\omega_o = \frac{2\pi}{T}$; where multiples of this frequency $n\omega_o$ are called harmonic frequencies

Conditions that ensure that f(t) can be expressed as a convergent Fourier series: (Dirichlet's conditions)

1. f(t) be single-values
2. f(t) have a finite number of discontinuities in the periodic interval
3. f(t) have a finite number of maxima and minima in the periodic interval
4. the integral $\int_{t_0}^{t_0+T} |f(t)|dt$; exists

These are sufficient conditions not necessary conditions; therefore if these conditions exist the functions can be expressed as a Fourier series. However if the conditions are not met the function may still be expressible as a Fourier series.

16.2 The Fourier Coefficients

Defining the Fourier coefficients:

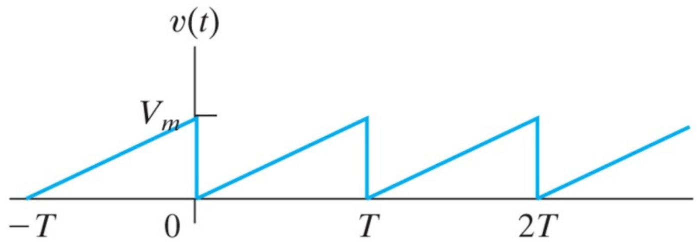
$$a_v = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)dt$$

$$a_k = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\cos k\omega_o t\, dt$$

$$b_k = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\sin k\omega_o t\, dt$$

Example 16.1

Find the Fourier series for the periodic waveform shown.



Assessment problems 16.1 & 16.2

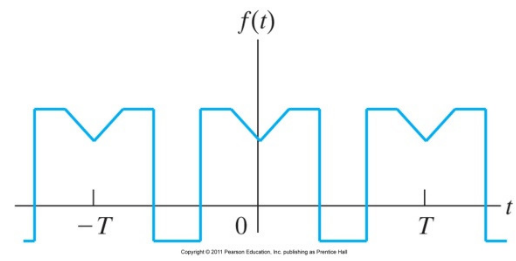
16.3 The Effects of Symmetry on the Fourier Coefficients

Four types of symmetry used to simplify Fourier analysis

1. Even-function symmetry
2. Odd-function symmetry
3. Half-wave symmetry
4. Quarter-wave symmetry

<u>Even-function symmetry</u>

$$f(t) = f(-t)$$



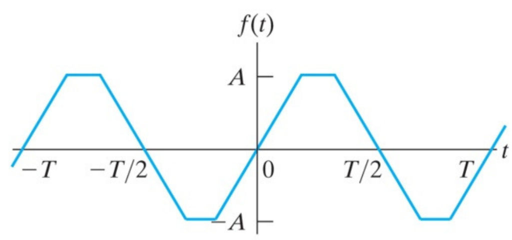
Simplified equations:

$$a_v = \frac{2}{T}\int_0^{\frac{T}{2}} f(t)dt$$

$$a_k = \frac{4}{T}\int_0^{\frac{T}{2}} f(t)\cos k\omega_o t\, dt \quad and \quad b_k = 0;\ for\ all\ k$$

<u>Odd-function symmetry</u>

$$f(t) = -f(-t)$$



Simplified equations:

$$b_k = \frac{4}{T}\int_{t_0}^{\frac{T}{2}} f(t)\sin k\omega_o t\, dt$$

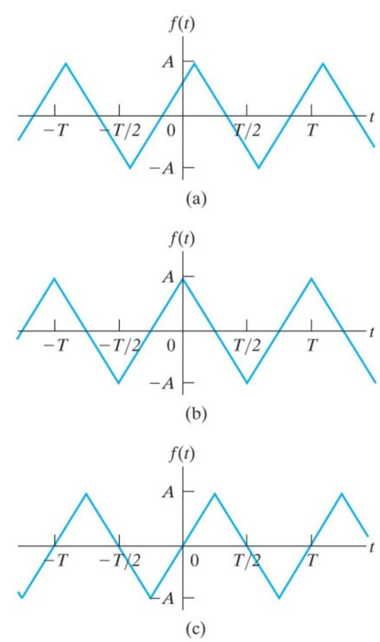
$$a_v = 0 \quad and \quad a_k = 0\ for\ all\ k$$

<u>Half-wave symmetry</u>

If the function is shifted one half period and inverted and look identical to the original then it is half-wave symmetric

$$f(t) = -f\left(t - \frac{T}{2}\right);$$

A half-wave symmetric function can be even, odd or neither.



Simplified Equations:

$$a_v = 0;\ a_k = 0\ and\ \ b_k = 0 \quad for\ even\ k$$

$$a_k = \frac{4}{T}\int_0^{\frac{T}{2}} f(t)\cos k\omega_o t\, dt \quad for\ odd\ k$$

$$b_k = \frac{4}{T}\int_{t_0}^{\frac{T}{2}} f(t)\sin k\omega_o t\, dt \quad for\ odd\ k$$

Quarter-wave symmetry

An expression that has both half-wave symmetry and even or odd symmetry

Half-wave & Even symmetry

$$a_k = \frac{8}{T}\int_0^{\frac{T}{4}} f(t)\cos k\omega_o t\,dt \quad for\ odd\ k$$

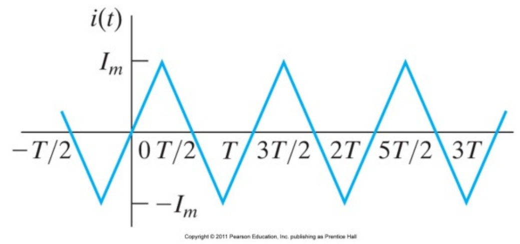
All other values are zero

Half-wave & Odd symmetry

$$b_k = \frac{8}{T}\int_{t_0}^{\frac{T}{4}} f(t)\sin k\omega_o t\,dt \quad for\ odd\ k$$

All other values are zero

Example 16.2



Assessment Problem 16.3

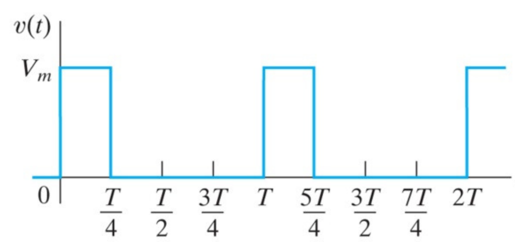
16.4 An Alternative Trigonometric Form of the Fourier Series

$$f(t) = a_v + \sum_{n=1}^{\infty} A_n \cos(n\omega_o t - \theta_n)$$

Where A_n and θ_n are a complex quantity

$$a_n - jb_n = \sqrt{a_n^2 + b_n^2}\angle\theta_n = A_n\angle - \theta_n$$

Example 16.3



Assessment Problem 16.4

16.5 Not Covered

16.6 Not Covered

16.7 Not Covered

16.8 The Exponential Form of the Fourier Series

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

Where

$$C_n = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)e^{-jn\omega_0 t}dt$$

Recalling

$$\cos n\omega_0 t = \frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2} \quad and \quad \sin n\omega_0 t = \frac{e^{jn\omega_0 t} - e^{-jn\omega_0 t}}{2j}$$

Euler's formula

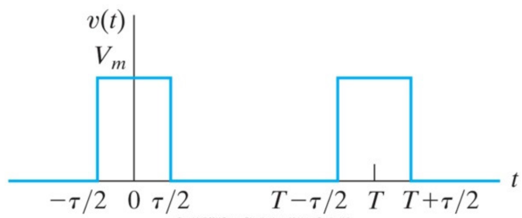
$$e^{jx} = \cos x + j\sin x \quad and \quad e^{-jx} = \cos x - j\sin x$$

Defining C_n

$$C_n = \frac{a_n - jb_n}{2} = \frac{A_n}{2}\angle -\theta_n \quad for \quad n = 1, 2, 3, \cdots$$

$$C_0 = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)dt = a_v$$

Example 16.6



Assessment Problem 16.8

16.9 Amplitude and Phase Spectra

<u>Amplitude spectrum:</u> the plot of the amplitude of each term of the Fourier series of f(t) versus frequency

<u>Phase spectrum:</u> the plot of the phase angle of each term versus frequency

<u>Line Spectra:</u> plots above; since they occur at discrete values of the frequency

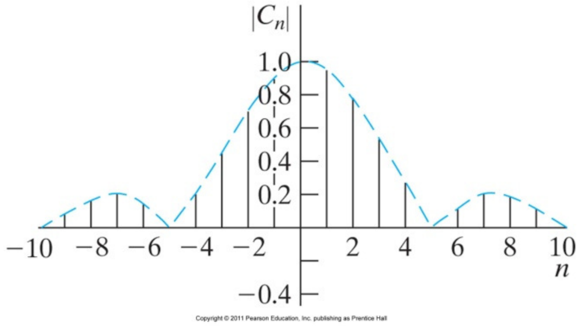
Illustration of Amplitude and Phase Spectra

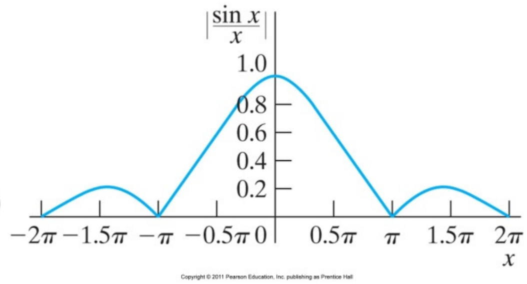
Referring to example 16.6

$$C_n = \frac{V_m \tau}{T}\frac{\sin\left(\frac{n\omega_0 \tau}{2}\right)}{\frac{n\omega_0 \tau}{2}}$$

Given $V_m = 5V$ and $\tau = \frac{T}{5}$

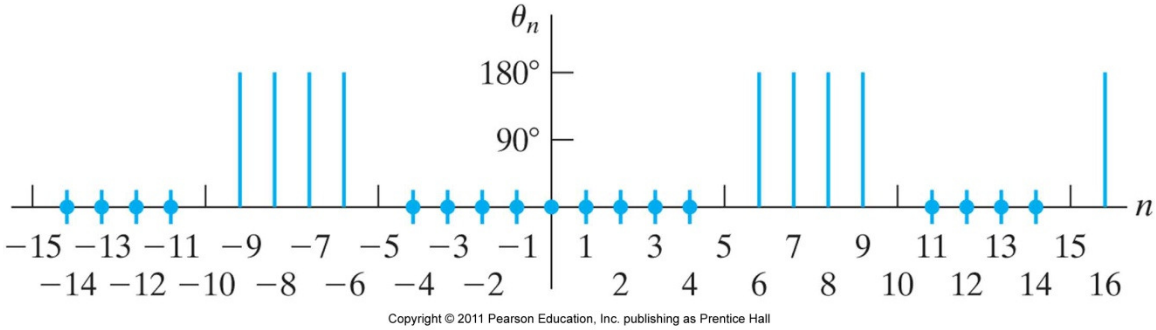
$$C_n = 1\frac{\sin\left(\frac{n\pi}{5}\right)}{\frac{n\pi}{5}}$$





Since the function is even: b_k=0,

$$a_n = \frac{4}{T}\int_0^{\frac{T}{2}} f(t)\cos k\omega_o t\,dt = \frac{4V_m}{T}\int_0^{\frac{\tau}{2}} \cos k\omega_o t\,dt = \frac{10}{k\pi}\sin\frac{\pi k}{5}$$



Effects of shifting f(t) on the time axis

Amplitude experiences no change

$$|C_n| = \left|C_n e^{-jn\omega_0 t_0}\right|$$

Phase is affected

$$\theta_n' = -\left(\theta_n + \frac{n\pi}{5}\right)$$