Chapter 7: Response of First-Order RL and RC Circuits

First-order circuits: circuits whose voltages and current can be described by first-order differential equations. (RL and RC circuits)

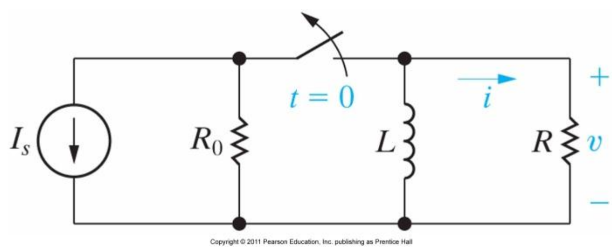
3-steps to analyzing

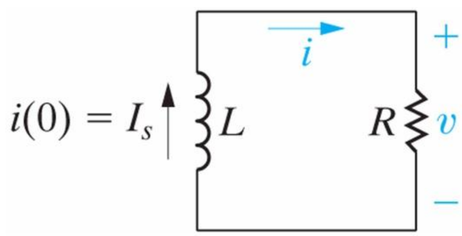
1. *Natural Response*: the currents and voltages that exist when stored energy is released to a circuit when the sources are abruptly removed
2. *Step Response:* currents and voltages that arise when energy is being stored due to the sudden application of a DC source.
3. Development of a general method to find the response of RL and RC circuits to any change of sources. (Norton and Thevenin Equivalents)

7.1 The Natural Response of an RL Circuit

Switch has been closed "a long time" all currents and voltages at constant values.

Opening the switch at t = 0





KVL around the circuit

$$L\frac{di}{dt} + Ri = 0$$

The highest order derivative is 1: hence first-order

Manipulating the equation

$$\frac{di}{dt}dt = -\frac{R}{L}idt \quad \rightarrow \quad \frac{di}{i} = -\frac{R}{L}dt \quad \rightarrow \quad \int_{i(t_0)}^{i(t)} \frac{dx}{x} = -\frac{R}{L}\int_{t_0}^{t} dt$$

$$ln\frac{i(t)}{i(0)} = -\frac{R}{L}t \quad \xrightarrow{yields} \quad i(t) = i(0)e^{-\frac{R}{L}t}$$

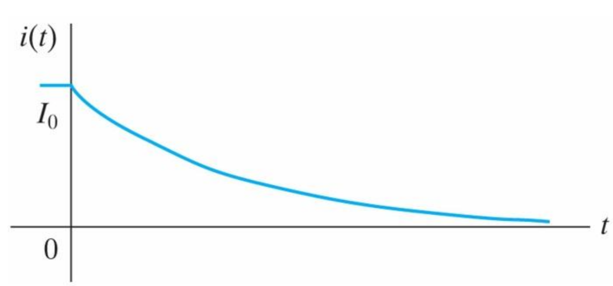
Assuming $i(0^-) = i(0^+) = I_0$

The Natural Response:

$$i(t) = I_0 e^{-\frac{R}{L}t}; \quad t \geq 0$$

And

$$v(t) = I_0 R e^{-\frac{R}{L}t}; \quad t \geq 0^+$$



$$p = {I_0}^2 R e^{-2\frac{R}{L}t}; \quad t \geq 0^+$$

$$w = \frac{1}{2}L{I_0}^2\left(1 - e^{-2\frac{R}{L}t}\right); \quad t \geq 0$$

Time constant (τ): the time it takes the natural response to decay by a factor of $^1/_e$

$$\tau = \frac{L}{R}$$

Rewriting the earlier equations

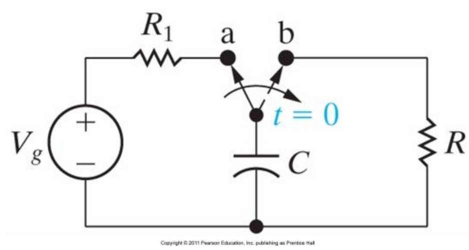
$$i(t) = I_0 e^{-\frac{t}{\tau}};\quad t \geq 0$$
$$v(t) = I_0 R e^{-\frac{t}{\tau}};\quad t \geq 0^+$$
$$p = {I_0}^2 R e^{-\frac{2t}{\tau}};\quad t \geq 0^+$$
$$w = \frac{1}{2} L {I_0}^2 \left(1 - e^{-\frac{2t}{\tau}}\right);\quad t \geq 0$$

Calculating the natural response of an RL circuit

- Find the initial current I_0 through the inductor
- Find the time constant
- Utilize $i(t) = I_0 e^{-\frac{t}{\tau}};\ t \geq 0$

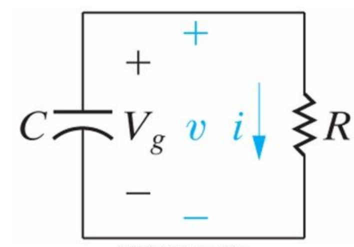
Review Examples 7.1 & 7.2 and Assessment Problems 7.1 & 7.2

7.2 The Natural Response of an RC Circuit



Again switch closed for "a long time" prior to opening.

$$C\frac{dv}{dt} + \frac{v}{R} = 0$$



$$v(t) = v(0) e^{-\frac{t}{RC}};\quad t \geq 0$$

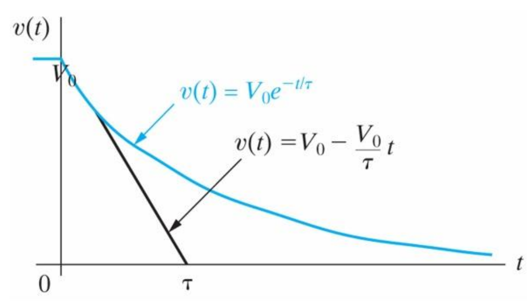
$$v(0^-) = v(0^+) = v(0) = V_g = V_0$$

The time constant

$$\tau = RC$$

Therefore

$$v(t) = V_0 e^{-\frac{t}{\tau}};\quad t \geq 0$$



$$i(t) = \frac{v(t)}{R} = \frac{V_0}{R} e^{-\frac{t}{\tau}};\quad t \geq 0^+$$
$$p = vi = \frac{{V_0}^2}{R} e^{-\frac{2t}{\tau}};\quad t \geq 0^+$$
$$w = \frac{1}{2} C {V_0}^2 \left(1 - e^{-\frac{2t}{\tau}}\right);\quad t \geq 0$$

Calculating the natural response of an RC circuit

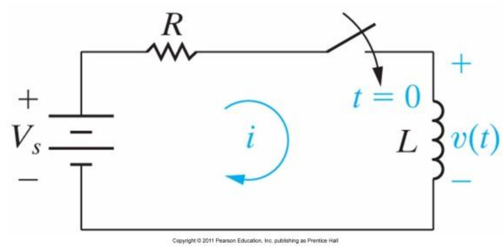
- Find the initial voltage V_0 across the capacitor

- Find the time constant
- Utilize $v(t) = V_0 e^{-\frac{t}{\tau}}$

Review Examples 7.3 & 7.4 and Assessment Problems 7.3 & 7.4

7.3 The Step Response of RL and RC Circuits

RL Circuit: Step Response



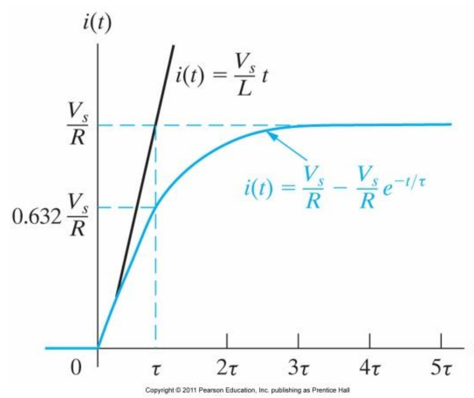
KVL (after the switch is closed)

$$V_s = iR + L\frac{di}{dt}$$

Utilizing separation of variables to solve for the step response (see book)

$$i(t) = \frac{V_s}{R} + \left(I_0 - \frac{V_s}{R}\right)e^{-\frac{R}{L}t}$$

As can be seen from the graph;



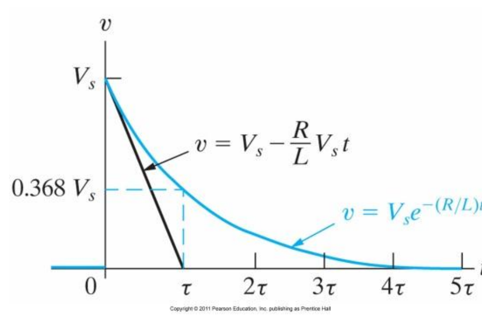
When the initial energy in the inductor I_0, is zero

$$i(t) = \frac{V_s}{R} - \frac{V_s}{R}e^{-\frac{R}{L}t}$$

The current will increase exponentially until it reaches its final value of ${}^{V_s}/_{R}$

$$v = L\frac{di}{dt} = L\left(-\frac{R}{L}\right)\left(I_0 - \frac{V_s}{R}\right)e^{-\frac{R}{L}t} = (V_s - I_0 R)e^{-\frac{R}{L}t}$$

As can be seen from the graph;



When the initial energy in the inductor I_0, is zero

$$v = V_s e^{-\frac{R}{L}t}$$

Initially zero, the voltage jumps to maximum $(V_s - I_0 R)$ decreasing exponentially to zero.

Alternative: Finding the voltage directly

$$i(t) = \frac{V_s}{R} - \frac{v(t)}{R}$$

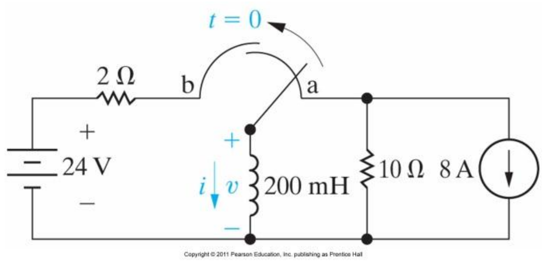
Differentiating both sides and multiplying by L

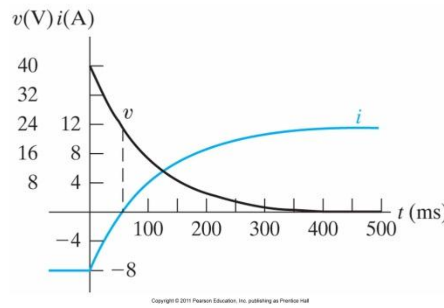
$$L\frac{di}{dt} = -\frac{L}{R}\frac{dv}{dt} = v \quad \rightarrow \frac{dv}{dt} + \frac{L}{R}v = 0$$

Note: The original KVL equation can be written in the same form

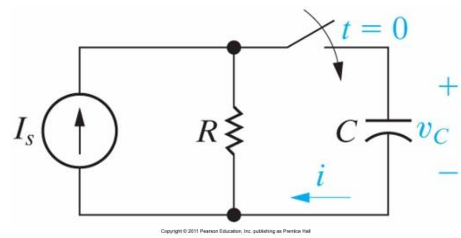
$$\frac{di}{dt} + i\frac{R}{L} = \frac{V_s}{L}$$

Review Example 7.5 and Assessment Problem 7.5





RC Circuit: Step response



KCL at the node

$$I_s = C\frac{dv_C}{dt} + \frac{v_C}{R} \quad \rightarrow \frac{dv_C}{dt} + \frac{v_C}{RC} = \frac{I_s}{C}$$

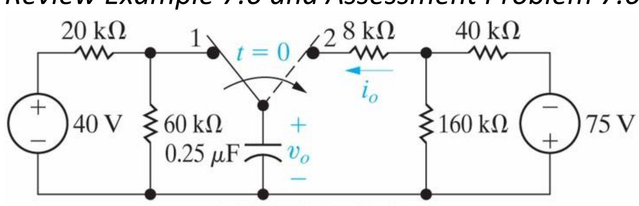
Again utilizing separation of variables method

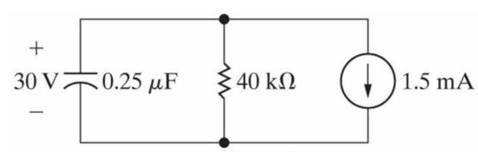
$$v_C = I_sR + (V_0 - I_sR)e^{-\frac{t}{RC}}; \qquad t \geq 0$$

Current equation

$$\frac{di}{dt} + \frac{1}{RC}i = 0 \ \rightarrow i = \left(I_s - \frac{V_0}{R}\right)e^{-\frac{t}{RC}}; \qquad t \geq 0^+$$

Review Example 7.6 and Assessment Problem 7.6

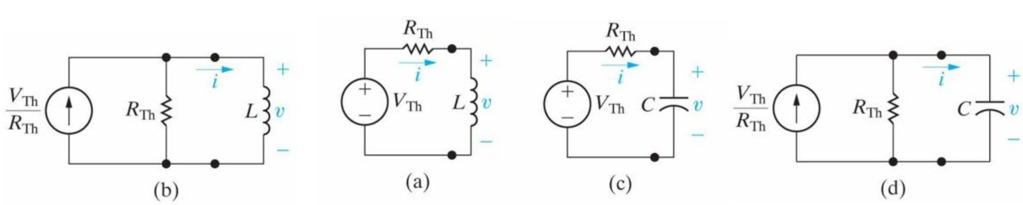




7.4 **A General Solution for the Step and Natural Responses**

Four possible first-order circuits:

a) An inductor connected to a Thévenin equivalent.
b) An inductor connected to a Norton equivalent.
c) A capacitor connected to a Thévenin equivalent.
d) A capacitor connected to a Norton equivalent.



General Solution

Let x equal the unknown for any of the circuits above and set up the differential equation in standard form

$$\frac{dx}{dt} + \frac{x}{\tau} = K$$

Where K is a constant that may be equal to zero

Since the circuits all have constant sources the final value of

$$\frac{dx}{dt} = 0$$

Therefore

$$x_f = K\tau$$

Where x_f is the final value of x.

Using separation of variables to solve for the first derivative

$$\frac{dx}{dt} = -\frac{x}{\tau} + K = \frac{-(x - K\tau)}{\tau} = \frac{-\left(x - x_f\right)}{\tau}$$

Rewriting

$$\frac{dx}{x - x_f} = \frac{-1}{\tau}dt \text{ intergrating both sides } \int_{x(t_0)}^{x(t)} \frac{du}{u - x_f} = \frac{-1}{\tau}\int_{x(t_0)}^{x(t)} dv$$

General solution for natural and Step response:

$$\boldsymbol{x(t) = x_f + \left[x(t_0) - x_f\right]e^{\frac{-(t-t_0)}{\tau}}}$$

The solution in words:

$$unknown\ variable = final\ value + [intial\ value - final\ value] \times e^{\frac{-[time - time\ of\ switching]}{time\ constant}}$$

Steps to calculating the natural or step response

1. Identify the variable of interest for the circuit. For RC circuits it is most convenient to choose the capacitive voltage; for RL circuits it is best to choose the inductive current.
2. Determine the initial value if the variable at $t_{0.}$ Note if you choose capacitive voltage or inductive current it is not necessary to distinguish between 0^- and 0^+ since they are continuous. Remember that the initial value is at 0^+
3. Calculate the final value of the variable at $t \rightarrow \infty$
4. Calculate the time constant for the circuit.

Review Example Problems 7.7 – 7.10

7.5 Sequential Switching

<u>Sequential switching</u>: whenever switching occurs more than once in a circuit.

Each circuit must be solved for the given state of the circuit at a given time and these values become the initial conditions for the next state.

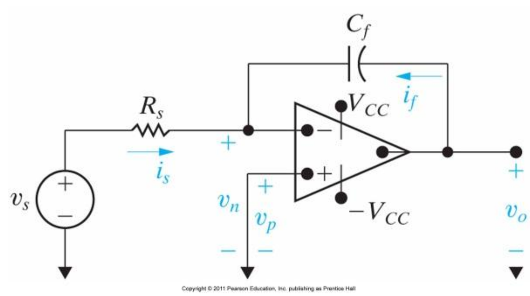
Review Example Problems 7.11 – 7.12 and Assessment Problems 7.7 & 7.8

7.6 Unbounded Response

<u>Unbounded response</u>: a circuit whose response grows rather decays exponentially with time. (Possible due to circuit containing a dependent source)

This may cause the R_{Th} for the circuit to become negative giving a negative time constant.

7.7 The Integrating Amplifier



For an ideal amplifier

$$i_p = i_n = 0 \text{ and } v_p = v_n$$

For the circuit

$$i_s + i_f = 0 \text{ and } v_p = v_n = 0$$

Therefor

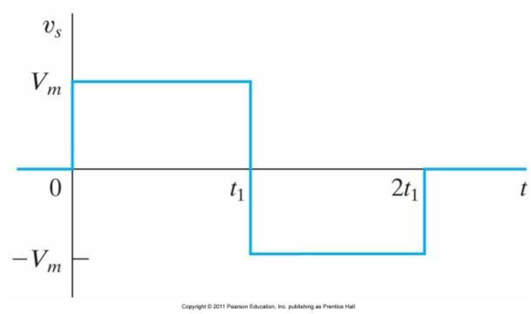
$$i_s = \frac{v_s}{R_s} \quad \text{and} \quad i_f = C_f \frac{dv_o}{dt}$$

Combining

$$\frac{dv_o}{dt} = \frac{1}{R_s C_f} v_s$$

Solving

$$v_o(t) = -\frac{1}{R_s C_f} \int_{t_0}^{t} v_s dy + v_o(t_0)$$



For the given input

$$v_o(t) = -\frac{1}{R_s C_f} V_m t + 0 \quad 0 \le t \le t_1$$

$$v_o(t) = \frac{V_m}{R_s C_f} t - \frac{2V_m}{R_s C_f} t_1 \quad t_1 \le t \le t_2$$