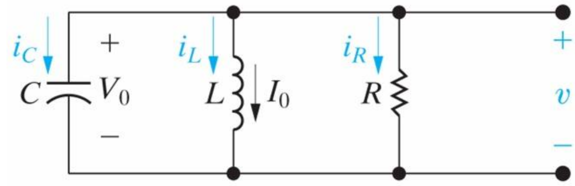
Chapter 8: Natural and Step Responses of the RLC Circuit

8.1 Introduction to the Natural Response of the Parallel RLC Circuit

Node Equation

$$\frac{v}{R} + \frac{1}{L}\int_0^t vD\tau + I_0 + C\frac{dv}{dt} = 0$$



Differentiating

$$\frac{1}{R}\frac{dv}{dt} + \frac{v}{L} + C\frac{d^2v}{dt^2} = \frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{v}{LC} = 0$$

General solution for a second-order differential

$$\text{Assuming} \quad V = Ae^{st} \quad \therefore \quad As^2e^{st} + \frac{As}{RC}e^{st} + \frac{Ae^{st}}{LC} = 0$$

$$Ae^{st}\left(s^2 + \frac{1}{RC}s + \frac{1}{LC}\right) = 0$$

For the equation to be zero; the general form is:

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$$

Solving for the roots

$$s_1 = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}} = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

And

$$s_2 = -\frac{1}{2RC} - \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}} = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

Where

$$\alpha = \frac{1}{2RC} \text{ is the Neper frequency}$$

$$\omega_0 = \frac{1}{\sqrt{LC}} \text{ is the resonant radian frequency}$$

TABLE 8.1 Natural Response Parameters of the Parallel *RLC* Circuit

Parameter	Terminology	Value In Natural Response
s_1, s_2	Characteristic roots	$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$ $s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$
α	Neper frequency	$\alpha = \frac{1}{2RC}$
ω_0	Resonant radian frequency	$\omega_0 = \frac{1}{\sqrt{LC}}$

Review Example 8.1 and Assessment Problem 8.1

8.2 The Forms of the Natural Response of the Parallel RLC Circuit

The Overdamped Voltage Response

When $\boldsymbol{\alpha^2 > \omega_0^2}$ (distinct and real roots)

<u>Overdamped:</u> the voltage or current approaches its final value without oscillation

The natural response:

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

The process for finding the response

1. Find the roots of the equation by solving for s_1 and s_2 for the values of R, L and C.
2. Find $v(0^+)$ and $\frac{dv(0^+)}{dt}$ using circuit analysis.

   Note: $v(0^+)$is the initial voltage in the capacitor & $\frac{dv(0^+)}{dt}$ relates to the initial inductor current where $i_c = C\frac{dv}{dt}$
3. Find A_1 and A_2 by solving the following equations simultaneously.

$$v(0^+) = A_1 + A_2 \quad \text{and} \quad \frac{dv(0^+)}{dt} = \frac{i_c(0^+)}{C} = s_1 A_1 + s_2 A_2$$

4. Substitute these values into the characteristic equation to get the expression for $v(t)$ for $t \geq 0$.

Review Examples 8.2 & 8.3 and Assessment Problems 8.2 & 8.3

The Underdamped Voltage Response

When $\boldsymbol{\omega_0^2 > \alpha^2}$ (complex roots)

<u>Underdamped:</u> the voltage or current oscillates around its final value

Given

$$s_1 = -\alpha + \sqrt{-(\alpha^2 - \omega_0^2)} = -\alpha + j\sqrt{\omega_0^2 - \alpha^2} = -\alpha + j\omega_d$$
$$s_2 = -\alpha - j\omega_d$$

Where ω_d is the damped radian frequency and equals $\sqrt{\omega_0^2 - \alpha^2}$.

The natural response:

$$v(t) = B_1 e^{-\alpha t}\cos\omega_d t + B_2 e^{-\alpha t}\sin\omega_d t$$

Follow the same steps in the process to obtain the response as in the overdamped example except for the equations in step 3 use:

$$v(0^+) = V_0 = B_1 \quad \text{and} \quad \frac{dv(0^+)}{dt} = \frac{i_c(0^+)}{C} = -\alpha B_1 + \omega_d B_2$$

<u>*Damping factor:*</u> (α) determines how quickly the oscillation will subside

Review Example 8.4 and Assessment Problem 8.4

The Critically Damped Voltage Response

When $\boldsymbol{\omega_0^2 = \alpha^2}$ or $\omega_0 = \alpha$

$$s_1 = s_2 = -\alpha = -\frac{1}{2RC}$$

The natural response:

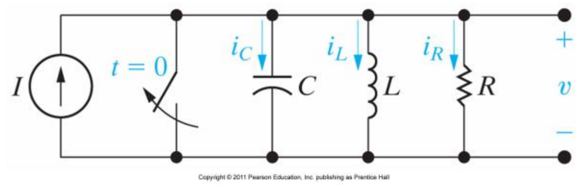
$$v(t) = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t}$$

Again, follow the same steps in the process to obtain the response except for the equations in step 3 use:

$$v(0^+) = V_0 = D_1 \text{ and } \frac{dv(0^+)}{dt} = \frac{i_c(0^+)}{C} = D_1 - \alpha D_2$$

Review Example 8.5 and Assessment Problem 8.5

8.3 The Step Response of the Parallel RLC Circuit



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$$i_L + \frac{v}{R} + C\frac{dv}{dt} = I$$

Where

$$v = L\frac{di_L}{dt} \text{ and } \frac{dv}{dt} = L\frac{d^2 i_L}{dt^2}$$

Simplifying

$$\frac{d^2 i_L}{dt^2} + \frac{1}{RC}\frac{di_L}{dt} + \frac{i_L}{LC} = \frac{I}{LC} = s^2 + \frac{1}{RC}s + \frac{1}{LC}$$

Since the equation is similar to the natural response with the exception of the source, the general form for the given response is

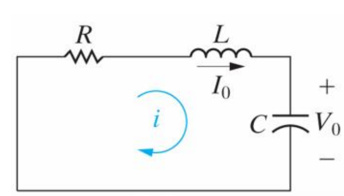
$$i = I_f + \{the\ form\ of\ the\ function\ for\ the\ natural\ response\}$$

$$v = V_f + \{the\ form\ of\ the\ function\ for\ the\ natural\ response\}$$

Where I_f and V_f represent the final values of the response function

Review Examples 8.5 – 8.10 and Assessment Problem 8.6

8.4 The Natural and Step Response of the Series RLC Circuit



Node Equation

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int_0^t iD\tau + V_0 = 0$$

Differentiating

$$R\frac{di}{dt} + L\frac{d^2 i}{dt^2} + \frac{i}{C} = \frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0$$

The general form is:

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

Solving for the roots

$$s_{1,2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

Where

$\alpha = \dfrac{R}{2L}$ is the Neper frequency; $\omega_0 = \dfrac{1}{\sqrt{LC}}$ is the resonant radian frequency

The current natural response for series RLC circuits:

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \text{ (overdamped)}$$
$$i(t) = B_1 e^{-\alpha t}\cos\omega_d t + B_2 e^{-\alpha t}\sin\omega_d t \text{ (underdamped)}$$
$$i(t) = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t} \text{ (critically damped)}$$

The voltage natural response for series RLC circuits:

$$v(t) = V_f + A'_1 e^{s_1 t} + A'_2 e^{s_2 t} \text{ (overdamped)}$$
$$v(t) = V_f + B'_1 e^{-\alpha t}\cos\omega_d t + B'_2 e^{-\alpha t}\sin\omega_d t \text{ (underdamped)}$$
$$v(t) = V_f + D'_1 t e^{-\alpha t} + D'_2 e^{-\alpha t} \text{ (critically damped)}$$

Review Examples 8.11 & 8.12 and Assessment Problems 8.7 & 8.8

Summarizing for Second Order circuit Responses

TABLE 8.2 The Response of a Second-Order Circuit is Overdamped, Underdamped, or Critically Damped

The Circuit is	When	Qualitative Nature of the Response
Overdamped	$\alpha^2 > \omega_0^2$	The voltage or current approaches its final value without oscillation
Underdamped	$\alpha^2 < \omega_0^2$	The voltage or current oscillates about its final value
Critically damped	$\alpha^2 = \omega_0^2$	The voltage or current is on the verge of oscillating about its final value

TABLE 8.3 In Determining the Natural Response of a Second-Order Circuit, We First Determine Whether it is Over-, Under-, or Critically Damped, and Then We Solve the Appropriate Equations

Damping	Natural Response Equations	Coefficient Equations
Overdamped	$x(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$	$x(0) = A_1 + A_2$; $dx/dt(0) = A_1 s_1 + A_2 s_2$
Underdamped	$x(t) = (B_1 \cos\omega_d t + B_2 \sin\omega_d t)e^{-\alpha t}$	$x(0) = B_1$; $dx/dt(0) = -\alpha B_1 + \omega_d B_2$, where $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$
Critically damped	$x(t) = (D_1 t + D_2)e^{-\alpha t}$	$x(0) = D_2$, $dx/dt(0) = D_1 - \alpha D_2$

TABLE 8.4 In Determining the Step Response of a Second-Order Circuit, We Apply the Appropriate Equations Depending on the Damping

Damping	Step Response Equations[a]	Coefficient Equations
Overdamped	$x(t) = X_f + A'_1 e^{s_1 t} + A'_2 e^{s_2 t}$	$x(0) = X_f + A'_1 + A'_2$; $dx/dt(0) = A'_1 s_1 + A'_2 s_2$
Underdamped	$x(t) = X_f + (B'_1 \cos\omega_d t + B'_2 \sin\omega_d t)e^{-\alpha t}$	$x(0) = X_f + B'_1$; $dx/dt(0) = -\alpha B'_1 + \omega_d B'_2$
Critically damped	$x(t) = X_f + D'_1 t e^{-\alpha t} + D'_2 e^{-\alpha t}$	$x(0) = X_f + D'_2$; $dx/dt(0) = D'_1 - \alpha D'_2$

[a] where X_f is the final value of $x(t)$.